combined tension and shear. However, more data and further studies are required before more refined relationships for various degrees of combined tension and shear loads can be developed.

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Statistical Identification of Structures

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A method is formulated for systematically using experimental measurements of the natural frequencies and mode shapes of a structure to modify stiffness and mass characteristics of a finite element model. Throughout the modification process, which does not require complete data, the finite element model remains consistent. An additional feature is that the engineer's confidence in the modeling of the various finite elements is quantified and incorporated into the revision procedure. Examples demonstrate the convergence and versatility of the method.

Introduction

A PRINCIPAL goal of the structural analyst is to formulate an analytical model of a structure which can be verified by actual test. Frequently, this model does not initially produce mode shapes and natural frequencies which concur with test results, and consequently an iterative cycle must be introduced to adjust the analytical model until the analysis and test results agree. The adjustment procedure is difficult and cannot be done adequately without the computer because of the large volume of computation necessary. In recent years a number of procedures have been developed, but none has received general acceptance although many have been successful for specific applications.[1]

Of the papers written on system identification, only a portion have presented methods which maintain the full order and physical significance of the mass and stiffness matrices of the structure.[2–6] Of these methods, several attempted to construct stiffness matrices from incomplete modal data and the remaining applied least squares techniques to obtain the desired property values. The use of least squares generally requires an overdetermined solution and consequently the amount of data obtained in the test must exceed in number the parameters in the structure which can be identified. As a result, the utilization of as much data as possible is very important. If only frequency is used, the number of identifiable parameters is severely limited and hence, if at all possible, measured modal deflections should be included in the procedure.

Data and modeling accuracies are both important considerations in the identification process. Certainly the accuracy of modal deflection measurements is not as good as the accuracy of frequency measurements. Therefore any procedure using least squares should include data accuracy in the weighting procedure. Modeling accuracy is more intuitive but still significant. In the development of the stiffness matrix, structural elements are included which can be modeled with varying confidence. Bending stiffnesses of uniform beams, for example, can be estimated much more accurately than bending stiffnesses for conical shells. Consequently those elements in the stiffness matrix relating to the beam have much less uncertainty associated with them than the stiffness elements relating to the conical shell. It is desirable for the method to be able to handle this variance of confidence.

The objective, therefore, of the work presented here is to propose a general method which will identify a finite element model of a structure capable of providing modal characteristics which are consistent with those measured in test. The method

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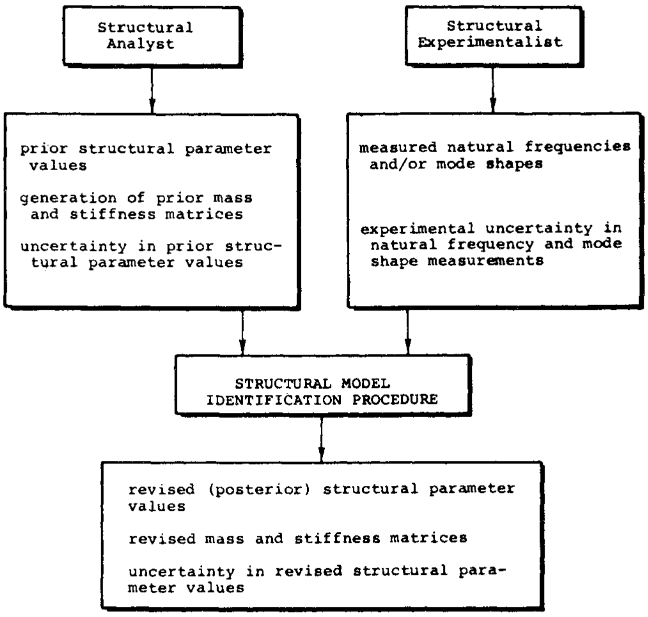

Fig. 1 Schematic of identification procedure.

maintains the specified finite element character of the model and uses values of the structural properties originally assigned to the model by the engineer as the starting point. The method then modifies original property values to make the modal characteristics conform to those observed in test. Accuracy of the values of the test data and the engineer's confidence in the values of the original analytic model properties are incorporated into the procedure.

The method helps bridge the gap between the structural analyst and the structural experimentalist. The former engineer establishes the prior analytical model and a description of his confidence in it. The latter provides experimental estimates of the natural frequencies and mode shapes plus a description of his confidence in such estimates. It is imperative to the success of the developed procedure that each engineer uses his professional judgement to provide realistic estimates of the confidence of the formulated data. Figure 1 shows a macro-schematic of the identification procedure.

Perturbation of the Eigenvalue-Eigenvector Solution about an Initial Estimate

The undamped natural frequencies and mode shapes of vibration are obtained from a solution of the structural dynamics eigenvalue problem. Inherent in this problem is the formulation of the mass and stiffness matrices of the structure. Once these matrices are formulated, the eigenvalues and eigenvectors of an n degree-of-freedom system are obtained by solving the characteristic equation

$$[K]\{x\}_i = \lambda_i [M]\{x\}_i \tag{1}$$

where $[K] = (n \times n)$ stiffness matrix, $[M] = (n \times n)$ mass matrix, $\lambda_i = i$th eigenvalue, $\omega_i = +(\lambda_i)^{1/2} = i$th natural frequency, and $\{x\}_i = (n \times 1)$ eigenvector (mode shape) of ith mode. The mass and stiffness matrices are functions of the structural parameters of the system and therefore the eigenvalues and mode shapes are implicit functions of these same parameters.

The functional relationship between the modal characteristics and the structural parameters is expressible in terms of a Taylor's series expansion.[7] It follows that

$$\begin{Bmatrix} \{\lambda\} \\ \{x\} \end{Bmatrix} = \begin{Bmatrix} \{\lambda(r_p)\} \\ \{x(r_p)\} \end{Bmatrix} + [T](\{r\} - \{r_p\}) \tag{2}$$

where $\{r\} = (m \times 1)$ *column matrix of structural parameters (e.g.,* EI, etc.); $\{r_p\} = (m \times 1)$ column matrix of prior estimates of structural parameters; $\{\lambda\}$ = column matrix of eigenvalues; $\{x\}$ = column matrix of modal displacements; $\{\lambda(r_p)\}$, $\{x(r_p)\}$ = column matrices of eigenvalues and modal displacements obtained from a solution to Eq. (1) with $\{r\} = \{r_p\}$; $[T]$ = eigenvalue and eigenvector partial derivative matrix, $[\partial(\lambda, x)/\partial(r)]$. $[T]$ can be expressed as the matrix decomposition

$$[T] = [B][A] \tag{3}$$

where

$$[B] = \begin{bmatrix} \left[\dfrac{\partial\lambda}{\partial k}\right] & \left[\dfrac{\partial\lambda}{\partial m}\right] \\ \left[\dfrac{\partial x}{\partial k}\right] & \left[\dfrac{\partial x}{\partial m}\right] \end{bmatrix} \tag{4}$$

and

$$[A] = \begin{bmatrix} \left[\dfrac{\partial k}{\partial r}\right] \\ \left[\dfrac{\partial m}{\partial r}\right] \end{bmatrix} \tag{5}$$

The individual submatrices of $[B]$ represent the partial derivatives of eigenvalues and modal displacements with respect to mass and stiffness coefficients. The submatrices are each arranged in the following format

$$\left[\frac{\partial\lambda}{\partial k}\right] = \begin{bmatrix} \dfrac{\partial\lambda_1}{\partial k_{11}} & \dfrac{\partial\lambda_1}{\partial k_{12}} & \dfrac{\partial\lambda_1}{\partial k_{13}} \cdots & \dfrac{\partial\lambda_1}{\partial k_{nn}} \\ \dfrac{\partial\lambda_2}{\partial k_{11}} & \dfrac{\partial\lambda_2}{\partial k_{12}} & \dfrac{\partial\lambda_2}{\partial k_{13}} \cdots & \dfrac{\partial\lambda_2}{\partial k_{nn}} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial\lambda_n}{\partial k_{11}} & \dfrac{\partial\lambda_n}{\partial k_{12}} & \dfrac{\partial\lambda_n}{\partial k_{13}} & \dfrac{\partial\lambda_n}{\partial k_{nn}} \end{bmatrix} \tag{6}$$

where k_{ij} is the ijth element in the stiffness matrix. Note that the size of each of the submatrices noted in Eq. (4) is determined by the number and location of the structural parameters as well as by the number of eigenvalue and modal displacements under consideration. In particular, the terms in this latter category are equal to the measured modal parameters. The calculation of the eigenvalue and modal displacement derivatives in the $[B]$ matrix requires additional comment.

The eigenvalue derivatives are[7,8]

$$\frac{\partial\lambda_i}{\partial k_{rs}} = \frac{x_{ri}\, x_{si}}{m_i^*} \tag{7}$$

and

$$\frac{\partial\lambda_i}{\partial m_{rs}} = -\lambda_i \frac{x_{ri}\, x_{si}}{m_i^*} \tag{8}$$

where

$$m_i^* = \{x\}_i^T [M]\{x\}_i \tag{9}$$

The ith eigenvalue derivative is only a function of the mass matrix and the ith mode shape.

Modal displacement derivatives may be calculated using any of three published methods. The first method[8] requires a full modal matrix and is therefore undesirable in large problems. The second[9-11] only requires a knowledge of the eigenvalue and eigenvector of the ith mode when the derivative of one of the ith modal displacements is sought. However, its disadvantages are: (1) it requires for each mode shape derivative the solution of a set of simultaneous equations equal in number to the order of the mass and stiffness matrices, and 2) the derivative of each element in the eigenvector must be calculated. The first item is a numerical handicap when the structural model involves many hundreds (or thousands) of degrees of freedom whereas the second item requires unnecessary calculations when only a small subset of the modal displacements are experimentally measured. The third method[12-14] expresses modal displacement derivatives as

$$\frac{\partial x_{ij}}{\partial k_{rs}} = \sum_{l=1}^{m} \left[\frac{x_{il} x_{rl} x_{sj}}{\lambda_j - \lambda_l} \right] (1 - \delta_{lj}) \tag{10}$$

and

$$\frac{\partial x_{ij}}{\partial m_{rs}} = \sum_{l=1}^{m} \left[\frac{(\delta_{lj} - 1)\lambda_j x_{il} x_{rl} x_{sj}}{(\lambda_j - \lambda_l)} - \frac{x_{il} x_{rl} x_{sj}}{2} \delta_{lj} \right] \tag{11}$$

where

$$\delta_{lj} = \begin{cases} 1 & l = j \\ 0 & l \neq j \end{cases} \tag{12}$$

Experience has shown that only a small number of terms are necessary in the summation for convergence to be obtained.[14] In all cases studied to date by the authors, good accuracy has been obtained by including in the summation two times the number of measured modes. However, the establishment of more general guidelines for the number of terms in the summation is a topic requiring additional research.

The $[A]$ matrix is used to relate the individual stiffness and mass elements to the structural parameters under consideration. This decomposition will enable the following procedure to establish estimates of individual structural parameters and maintain the internal consistency of the finite element model. Most present identification procedures provide revised mass and stiffness matrices which usually have no physical meaning. The general form of the submatrices in $[A]$ is evidenced by the matrix

$$\left[\frac{\partial k}{\partial r}\right] \equiv \begin{bmatrix} \frac{\partial k_{11}}{\partial r_1} & \frac{\partial k_{11}}{\partial r_2} & \cdots & \frac{\partial k_{11}}{\partial r_m} \\ \frac{\partial k_{12}}{\partial r_1} & \frac{\partial k_{12}}{\partial r_2} & \cdots & \frac{\partial k_{12}}{\partial r_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial k_{nn}}{\partial r_1} & \frac{\partial k_{nn}}{\partial r_2} & \cdots & \frac{\partial k_{nn}}{\partial r_m} \end{bmatrix} \tag{13}$$

An example of the $[A]$ matrix is shown for a two element beam in the example section of this paper.

Derivation of the Statistical System Identification Equations

The preceding section developed an expression which related the eigenvalues and modal displacements to the system structural parameters using a Taylor's series expansion. In this section, the relationship is used to solve the inverse problem of establishing a value for the system structural parameters given the eigenvalues and modal displacements. The procedure used in the derivation is based upon minimizing the variance associated with the estimator.[15]

Equation (2) is now written in the format

$$\{Y\} = [T]\{R\} \tag{14}$$

where

$$\{Y\} = \begin{Bmatrix} \{\lambda - \lambda_p\} \\ \cdots \\ \{x - x_p\} \end{Bmatrix} \tag{15}$$

and

$$\{R\} = \{r - r_p\} \tag{16}$$

with the subscripts $(\ )_p$ denoting the eigenvalues and eigenvectors of the system when the properties are $\{r\} = \{r_p\}$.

Before proceeding let us establish the statistical characteristics of the random variables in Eq. (14). Prior to when the test data are available, we can assume that the unknown structural parameters $\{r\}$ are normally distributed with mean

$$\{\mu_r\} = E[\{r\}] = \{r_p\} \tag{17}$$

The covariance matrix of $\{r\}$, which must be specified by the structural analyst, is

$$[S_{rr}] = E[\{r - r_p\}\{r - r_p\}^T] \tag{18}$$

Using Eq. (16), the mean and covariance matrix of $\{R\}$ are

$$\{\mu_R\} = E[\{R\}] = E[\{r - r_p\}] = \{0\} \tag{19}$$

and

$$[S_{RR}] = E[\{R\}\{R\}^T] = [S_{rr}] \tag{20}$$

In Eq. (14), if $\{R\}$ is described by a multivariate normal distribution, then $\{Y\}$ through the transformation is also normally distributed with mean and covariance

$$\{\mu_Y\} = E[\{Y\}] = \{0\} \tag{21}$$

and

$$E[\{Y\}\{Y\}^T] = E[[T]\{R\}\{R\}^T[T]^T] = [T][S_{rr}][T]^T \tag{22}$$

The zero mean in Eq. (20) implies

$$E\left[\begin{Bmatrix} \{\lambda\} \\ \cdots \\ \{x\} \end{Bmatrix}\right] = \begin{Bmatrix} \{\lambda_p\} \\ \cdots \\ \{x_p\} \end{Bmatrix} \tag{23}$$

Equation (14) could be used directly in developing an estimator for $\{R\}$ if the measurements, $\{Y\}$, were made without error. However, the measurements will contain errors which are independent of the uncertainty leading from the structural parameters $\{r\}$, but are not described in Eq. (22). To account for this, a vector $\{\varepsilon\}$ is added to Eq. (14) to account for the measurement error. The vector $\{\varepsilon\}$ has a zero mean and a covariance $[S_{\varepsilon\varepsilon}]$ specified from the test measurement errors. The linear relationship in Eq. (14) now becomes

$$\{Y\} = [T]\{R\} + \{\varepsilon\} \tag{24}$$

The mean and covariance matrices for $\{Y\}$ are now

$$E[\{Y\}] = \{0\}$$

and

$$[S_{YY}] = E[\{Y\}\{Y\}^T] = E[([T]\{R\} + \{\varepsilon\})([T]\{R\} + \{\varepsilon\})^T] = [T][S_{rr}][T]^T + [S_{\varepsilon\varepsilon}] \tag{25}$$

The covariance of $\{Y\}$ and $\{R\}$, which will be needed later in the development is

$$[S_{YR}] = E[\{Y\}\{R\}^T] = E[([T]\{R\} + \{\varepsilon\})\{R\}^T] = E[[T]\{R\}\{R\}^T + \{\varepsilon\}\{R\}^T] = [T][S_{RR}] = [T][S_{rr}] \tag{26}$$

where $E[\{\varepsilon\}\{R\}^T] = [0]$ because the vectors $\{\varepsilon\}$ and $\{R\}$ are statistically independent.

The objective of this procedure is to find a best linear unbiased estimator of $\{R\}$ based on measured values of $\{\lambda\}$ and $\{x\}$ and prior estimates of the structural parameters $\{r\}$. We are therefore seeking an equation of the form

$$\{R^*\} = [G]\{Y\} \tag{27}$$

The matrix $[G]$ must now be defined in such a way as to minimize the variance of the difference between the true value of $\{R\}$ and the estimated value denoted by $\{R^*\}$. That is, the equation

$$[Q] = E[\{R^* - R\}\{R^* - R\}^T] \tag{28}$$

is minimized with respect to the selection of $[G]$. To find the unique values of the elements of $[G]$ which minimize $[Q]$, take the variation of $[Q]$ with respect to $[G]$ and set it equal to zero. First substitute Eqs. (27) into Eq. (28) and obtain

$$[Q] = E[([G]\{Y\} - \{R\})([G]\{Y\} - \{R\})^T] = [G][S_{YY}][G]^T - [G][S_{YR}] - [S_{YR}]^T[G]^T + [S_{RR}] \tag{29}$$

Then taking the variation of this with respect to $[G]$ we obtain

$$0 = [\delta G]([S_{YY}][G]^T - [S_{YR}]) + ([G][S_{YY}] - [S_{YR}]^T)[\delta G]^T \tag{30}$$

Thus

$$[G] = [S_{YR}]^T[S_{YY}]^{-1} \tag{31}$$

and

$$\{R^*\} = [S_{YR}]^T[S_{YY}]^{-1}\{Y\} \tag{32}$$

which after substitution from Eqs. (15, 16, 25, and 26) becomes

$$\{r^*\} = \{r_p\} + [S_{rr}][T]^T([T][S_{rr}][T]^T + [S_{\varepsilon\varepsilon}])^{-1} \times \left(\begin{Bmatrix} \lambda \\ \cdots \\ x \end{Bmatrix} - \begin{Bmatrix} \lambda_p \\ \cdots \\ x_p \end{Bmatrix}\right) \tag{33}$$

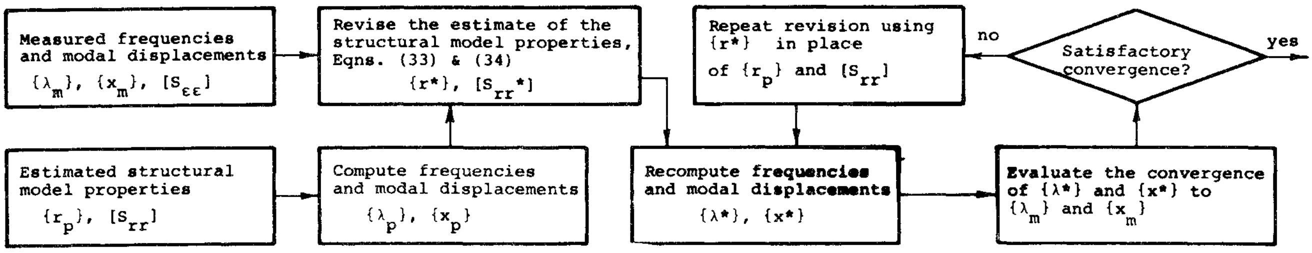

Fig. 2 The operations in revising the structural property estimates.

The covariance of the estimate of $\{r^*\}$ is

$$[S_{rr}^*] = E[\{R^* - R\}\{R^* - R\}^T]$$
$$= [S_{RR}] - [S_{YR}]^T[S_{YY}]^{-1}[S_{YR}]$$
$$= [S_{rr}] - [S_{rr}][T]^T([T][S_{rr}][T]^T + [S_{\varepsilon\varepsilon}])^{-1}[T][S_{rr}] \quad (34)$$

Implementation

Figure 2 describes the operations in the estimation procedure. If the process were truly linear, the method would converge to the best estimate in one step. However, perturbations of the eigenvalue relations are usually nonlinear and consequently a solution must be obtained by repeated iterations. At the end of each iteration the convergence must be evaluated and if not satisfactory the steps must be repeated, each time using $\{r^*\}$ from the last revision along with the original $\{\lambda_m\}$, $\{x_m\}$, $[S_{\varepsilon\varepsilon}]$, and $[S_{rr}]$.

The following points are relevant regarding the procedure:

1) The method operates quite satisfactorily with incomplete data. That is, not all natural frequencies or modal displacements are required to obtain an estimate. In fact, an answer can be obtained using a single data point (an underdetermined solution) although quite obviously a better solution is obtained with a larger quantity of data.

2) In each cycle the eigenvalues/vectors are recomputed and new derivative matrices are computed. This permits the reference point for the estimation procedure to shift and accommodates for nonlinearity in the problem.

3) The $[S_{rr}^*]$ obtained in the solution is a measure of the quality of the estimate of the structural properties $\{r\}$. The diagonal elements of $[S_{rr}^*]$ are the variances, i.e., $\sigma_{r_i}^2$, of each of the structural property estimates. Therefore the method shows where the confidence in the new property values is the greatest and the least.

4) The method will usually converge within a few cycles with the largest step being the first. Convergence can be based on whether the values of $\{\lambda^*\}$ and $\{x^*\}$ from the revision process fall within prescribed confidence bounds (typically 95%) of $\{\lambda_m\}$ and $\{x_m\}$. The confidence bounds are based on the measurement accuracies of $\{\lambda_m\}$ and $\{x_m\}$ which are contained in the covariance matrix $[S_{\varepsilon\varepsilon}]$. Note, $[S_{\varepsilon\varepsilon}]$ is usually a diagonal matrix because the measurement errors are assumed to be statistically independent.

5) Once $\{\lambda_m\}$, $\{x_m\}$, $[S_{\varepsilon\varepsilon}]$, $\{r_p\}$ and $[S_{rr}]$ are established, the method converges to a unique solution. Obviously the solution of the problem will be different if the prior estimates $\{r_p\}$ and $[S_{rr}]$ vary or if the quantity of data is changed.

6) The convergence of the results to the measured data will depend upon the accuracy of the measurements. A situation where data accuracy is very poor and modeling error is assumed to be small will result in very little revision to the structural properties. On the other hand if the data accuracy is excellent and acknowledged by small values in $[S_{\varepsilon\varepsilon}]$, the method will converge to yield answers consistent with measurements.

7) The nonlinearities inherent in the problem, will cause difficulty if the measured data have more error than expressed by $[S_{\varepsilon\varepsilon}]$. Therefore it is important to be somewhat conservative in estimating $[S_{\varepsilon\varepsilon}]$ and make allowance for error due to the existence of damping and the nonorthogonality of modes.

The uncertainties, $[S_{rr}]$ and $[S_{\varepsilon\varepsilon}]$, are not always easy to estimate. Measurement accuracies should be based on the scaling accuracy and, as mentioned before, consideration should be given to the fact that a measured mode will diverge from a normal mode. Conservatism in establishing $[S_{rr}]$ is not quite so critical, but it is very important to establish relative accuracy of the parameters. In the introduction it was mentioned that it is easier to estimate bending of a beam than bending of a conical shell. The relative uncertainties of these estimates fit into $[S_{rr}]$ which in turn influences the revision process.

Beam Example

The theory developed in the preceding sections of this Paper will now be applied to a small example problem. Figure 3 shows a free-free beam which is modeled using two finite elements and six generalized coordinates. The system contains two rigid body modes and four elastic modes. The random structural parameters are EI_1 and EI_2 where E = bending modulus of elasticity, and I_j = cross-sectional moment of inertia of element j. Therefore, in terms of the previously defined notation

$$r_1 = EI_1 \qquad r_2 = EI_2$$

The $[A]$ matrix for this sample problem is shown in Eq. (35).

$$[A] = \left(\frac{1}{L^3}\right)\begin{bmatrix} 12 & 0 \\ 6L & 0 \\ -12 & 0 \\ 6L & 0 \\ 4L^2 & 0 \\ -6L & 0 \\ 2L^2 & 0 \\ 12 & 12 \\ -6L & 6L \\ 0 & -12 \\ 0 & 6L \\ 4L^2 & 4L^2 \\ 0 & -6L \\ 0 & 2L^2 \\ 0 & 12 \\ 0 & 6L \\ 0 & 4L^2 \end{bmatrix} \begin{matrix} \leftarrow k_{11} \\ k_{12} \\ k_{13} \\ k_{14} \\ k_{22} \\ k_{23} \\ k_{24} \\ k_{33} \\ k_{34} \\ k_{35} \\ k_{36} \\ k_{44} \\ k_{45} \\ k_{46} \\ k_{55} \\ k_{56} \\ \leftarrow k_{66} \end{matrix} \quad (35)$$
$$\qquad \uparrow \quad \uparrow$$
$$\quad (EI_1) \quad (EI_2)$$

Where k_{ij} are the addresses of the elements in the 6×6 stiffness matrix.

A series of case studies were conducted on the free-free beam. In these studies beam mass and length values were selected as constants and target values for (EI_1) and (EI_2) established as (2.450×10^{10}) and (2.550×10^{10}), respectively. With these values the free-free natural frequencies and mode shapes were evaluated. The first two natural frequencies of vibration were 43.99 Hz and 2341 Hz, respectively. These target values

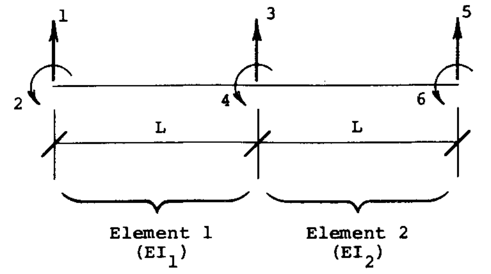

Fig. 3 Free-free beam example.

were then imagined to be the true values. The experimentally measured modal values were assumed to be identical with the previously calculated values.

Then, an original analytical model was postulated to be different from the above. The original model possessed the same length and mass properties as the above model but its elastic structural parameters were selected to be $EI_1 = 2.664\ 10^{10}$ and $EI_2 = 2.964\ 10^{10}$. These parameter values yielded values for the first two analytical natural frequencies of 46.187 Hz and 2494 Hz.

Table 1 shows results obtained for several case studies using the original analytical model. Each case utilized different quantities of experimental data. It is apparent that with the exception of the first case all identified values were acceptable. The values noted were obtained after one iteration. The values after the second iteration are shown in parentheses.

Table 1 Identification of a two-element beam

		Identified values			
Case	Description of measured data	$(EI_1/10^{10})$	$(EI_2/10^{10})$	f_1, Hz	f_2, Hz
	Target values	2.450	2.550	43.99	2341
	Original analytical values (prior)	2.664	2.964	46.87	2494
1)	First natural frequency only	2.297 (2.299)	2.668 (2.669)	43.98 (43.99)	2338 (2339)
2)	First two natural frequencies	2.406 (2.412)	2.534 (2.533)	43.97 (44.00)	2340 (2341)
3)	First two natural frequencies plus translational modal displacements in first two modes	2.406 (2.412)	2.534 (2.533)	43.97 (44.00)	2340 (2341)
4)	First two natural frequencies plus all modal displacements in first two modes	2.415 (2.424)	2.521 (2.519)	43.96 (43.99)	2339 (2341)

Saturn V Example

A lateral vibration model of Saturn V was chosen to demonstrate the procedure on a larger problem. The model is shown in Fig. 4. It was broken into 28 beam elements having both bending and shear stiffness. The random bending (EI) and random shear (GA_s) stiffness in each element summed to a total of 56 structural parameters (r_i) which could be revised in the identification procedure. Figure 4 also shows the percentage uncertainty in the stiffness estimates along the vehicle.

Test data were obtained for the first three elastic modes. Lateral modal deflections were available at 19 stations along the vehicle. The deflection data for the three modes plus the measured frequency of each mode provided 60 measurements to be used in revising the 56 parameters.

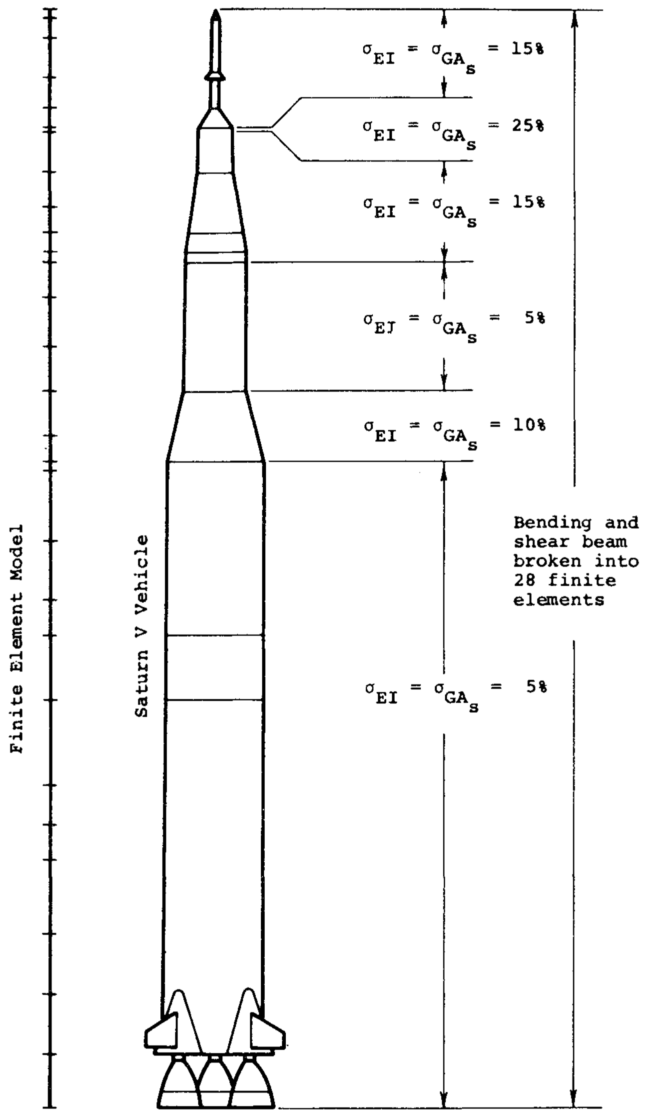

Fig. 4 Saturn V vibration model.

In this particular case the measured frequencies were less than 5% different from the predicted ones to start with and the identification procedure produced convergence to less than 0.03% in three iterations, see Table 2. In the process, stiffnesses throughout the model were revised upward and downward by percentages varying from zero to 48.3, see Fig. 5. The largest

Table 2 Saturn V results

	First natural freq. (Hz)	% Diff from test	Second natural freq. (Hz)	% Diff from test	Third natural freq. (Hz)	% Diff from test
Test values	1.106	...	1.821	...	2.547	...
Prior estimate (finite element model)	1.056	4.52	1.719	5.60	2.578	1.21
1st iteration	1.101	0.45	1.811	0.54	2.555	0.31
2nd iteration	1.106	0.0	1.821	0.0	2.550	0.11
3rd iteration	1.106	0.0	1.822	0.05	2.548	0.03
4th iteration	1.106	0.0	1.821	0.0	2.548	0.03
5th iteration	1.106	0.0	1.821	0.0	2.548	0.03

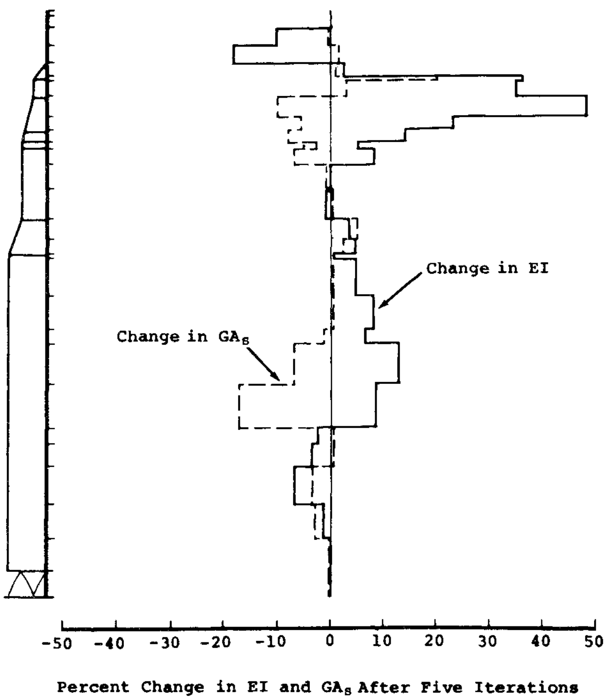

Fig. 5 *EI* **and** GA_s **modifications in the Saturn V vehicle using the revision procedure.**

shifts in stiffness were the bending stiffnesses in the regions of highest uncertainty in Fig. 4.

The $[S_{rr}]$ matrix used in this case was a 56×56 diagonal matrix with each element on the diagonal representing the estimated variance (σ^2) of a bending or shear stiffness property of one of the finite elements (see Fig. 4). The $[S_{\varepsilon\varepsilon}]$ matrix was also diagonal with each element representing the variance (σ^2) of a measurement. The $[S_{\varepsilon\varepsilon}]$ matrix was of dimension 60×60. Each element is the variance (standard deviation squared) of the error in one element of the measurement error vector (ε). The standard deviation of all frequency measurement errors was chosen to be 0.2% of their mean (measured) values. The corresponding variances of the eigenvalue errors were determined accordingly. The standard deviation of all modal displacement measurement errors was chosen to be 0.2% of the largest mean displacement in each respective mode. This particular run (five iterations) used approximately 40 sec of UNIVAC 1108, Exec 8 time. The major cost was in the repeated eigenvalue/vector computation. The cost for larger problems will grow proportionately with the eigenvalue cost.

As mentioned earlier, the only test for convergence is whether the frequencies and mode shapes from the revised model match those measures in test. In this case the revised frequencies match very well and 82% of the displacement points (corresponding to the test points) in the revised modes fell within the 90% confidence bounds of the measured modes. This last statement implies that after five iterations the revised model had not quite reached a point of providing modes which conform with the test. This may occur frequently because the type of analytical model may never conform exactly to the true structure (is a beam model adequate?) or because the measured modes are not true normal modes.

Conclusions

A method for the statistical identification of a structure has been formulated and demonstrated. It uses experimental measurements of natural frequencies and mode shapes to modify the structural parameters of a finite element model. The method preserves the consistency of the model.

The examples presented, while limited in total scope, demonstrate that the statistical identification method can be an effective tool in structural dynamics.

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